Weighting and comparison robustness with composite indicators

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- Examples abound:
  - Human Development Index (UNDP)
  - Ease of Doing Business Index (World Bank)
  - Environmental Sustainability Index (WEF)
  - Index of Economic Freedom (Heritage Foundation)
  - Global Peace Index (Vision of Humanity)
  - Child Well-being Index (UNICEF)
- Handbook on constructing composite indicators (JRC and OECD 2008)

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- A key moving part is the component weights

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  - Cahill (2005); Saisana et al. (2005); Nardo et al. (2008); Cherchye et al. (2008a,b); Foster et al. (2009, 2012, 2013); Permanyer (2011); Zheng and Zheng (2015); Seth and McGillivray (2018)

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  - Instead of replacing equal weights, advocate tests for robustness of rankings to a set of alternative weights (also broader sensitivity analyses)

Today's Presentation

- Draws from
  - Foster JE, McGillivray M, Seth S (2009) Rank robustness of composite indices. OPHI Working paper 26, University of Oxford
  - Foster JE, McGillivray M, Seth S (2013) Composite indices: Rank robustness, statistical association and redundancy. Econometric Reviews 32:35–56
  - Seth S, McGillivray M (2018) Composite indices, alternative weights, and comparison robustness. Social Choice and Welfare. 51:657–679
- A gap in the literature
  - Lack of appropriate normative framework for selecting such a set of alternative weights

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- w: A d-dimensional weight vector
- $\mathcal{W}$: Set of all possible weight vectors

Composite Indices and Comparison Robustness

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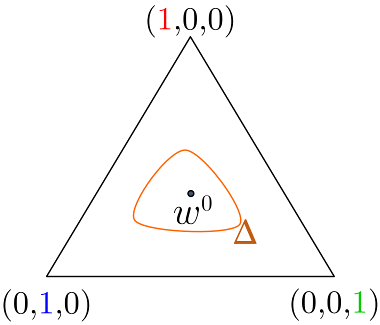
- $C(y; w^0) \geq C(x; w^0)$: Performance vector y has equal or higher composite index value than x at w_0 ($y \ C_0 \ x$)
- Comparison $y \ C_0 \ x$ is robust with respect to a set of alternative weights Δ if and only if $C(y; w) \geq C(x; w)$ for all weights in Δ (Foster et al. 2009, 2013)

Set of Alternative Weights (three dimensions)



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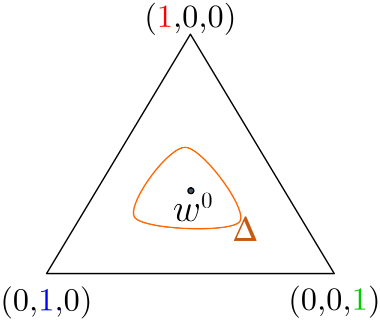

- Permanyer (2011): Did not propose particular set
- Foster et al. (2013): ε-contamination model
- Zheng and Zheng (2015): $\Delta = \mathcal{W}$ (entire simplex)

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  - Δ is convex hull of unique permutations of the most unequal weighting vector $\bar{w} \in \Delta$

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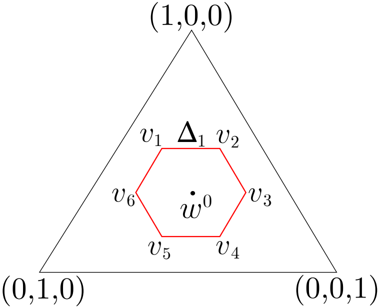

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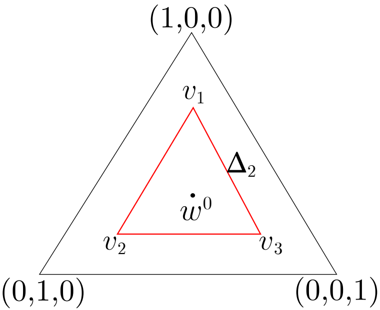

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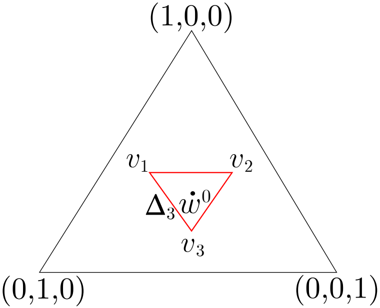

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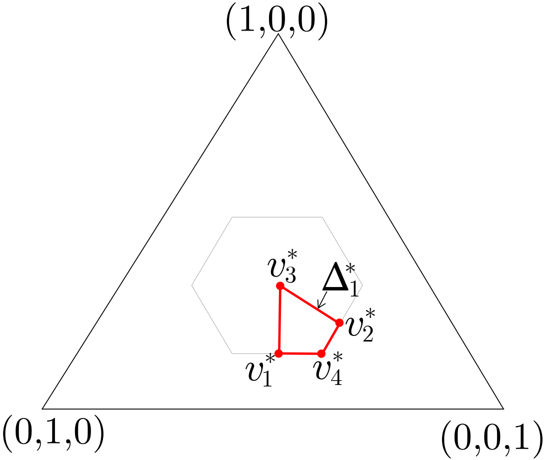
  2. Suppose, $D = 3$, $0.1 \leq w_1 \leq 0.4$, $0.25 \leq w_2 \leq 0.45$ and $0.3 \leq w_3 \leq 0.7$

Example: $\alpha = 1/6$; $\beta = 1/2$; $w_1 \leq w_2 \leq w_3$



Vertices:

$v_1^* = (1/6, 5/12, 5/12)$
$v_2^* = (1/4, 1/4, 1/2)$
$v_3^* = (1/3, 1/3, 1/3)$
$v_4^* = (1/6, 1/3, 1/2)$

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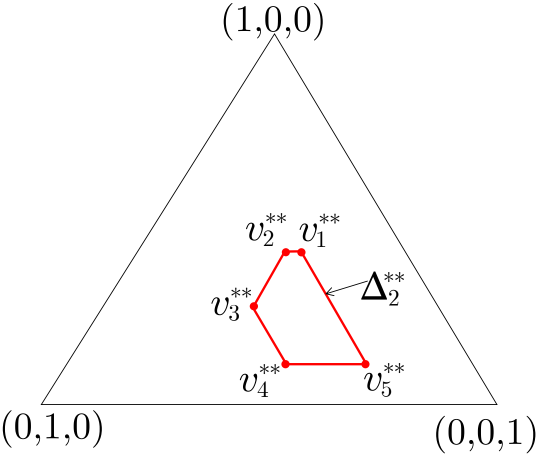

Vertices:

$v_1^{**} = (0.40, 0.25, 0.35)$
$v_2^{**} = (0.40, 0.30, 0.30)$
$v_3^{**} = (0.25, 0.45, 0.30)$
$v_4^{**} = (0.10, 0.45, 0.45)$
$v_5^{**} = (0.10, 0.25, 0.65)$

Robustness of Pair-wise HDI Comparisons

- A number of studies have questioned equal weights and analysed the robustness of HDI comparisons
  - Kelley (1991): argued for higher weight on income, but acknowledged difficulty
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  - Cahill (2005): six alternative weighing schemes yielded similar ranking
  - Cherchye et al. (2008): 75% pair-wise comparisons in 2002 not robust (subject to alternative normalizations, aggregation methods, and weights)
  - Foster et al. (2009): 70% pair-wise comparisons fully robust in 1998 and 2004
  - Zheng and Zheng (2015): 7 of the 45 pair-wise comparisons (among top 10) were robust in 2014

Illustration: Robust Changes in the HDI

- How robust are inter-temporal changes in the Human Development Index?
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  - Arithmetic mean: $HDI_A = \frac{1}{3}\sum_{d=1}^{3} w_d x_d$
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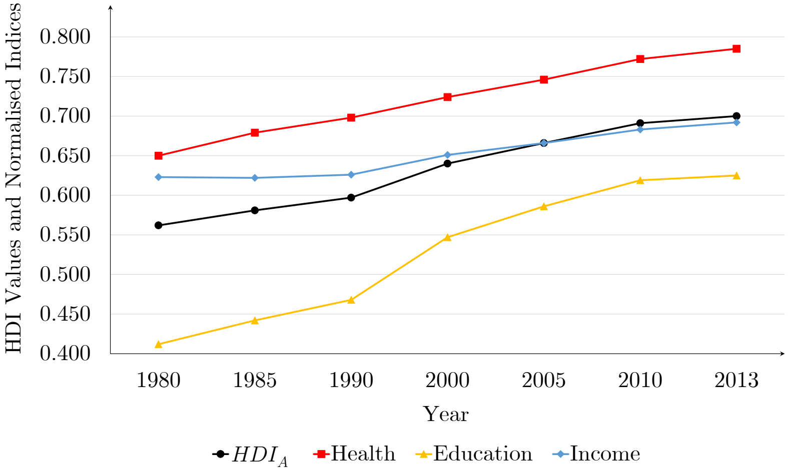
- Data for all component indices were available for 123 countries (UNDP website)

Improvements in HDI_A and Component Indices



Robust Changes in HDI_A and HDI_G over Time

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Time Period	Change in HDI_A				Change in HDI_G			
	Increase	Robust	Decrease	Robust	Increase	Robust	Decrease	Robust
1980–85	111	81	12	1	116	83	7	1
1985–90	106	81	17	5	108	87	15	5
1990–00	110	101	13	1	112	100	11	1
2000–05	117	109	6	0	116	107	7	0
2005–10	121	110	2	0	122	108	1	0
2010–13	113	99	10	0	113	99	10	0

Source: Author computations using UNDP data.

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	Change in HDI_A		Change in HDI_G	
Number of Time Periods	Number of Robust Increases	Number of Robust Decreases	Number of Robust Increases	Number of Robust Decreases
6	36	0	38	0
5	37	0	35	0
4	36	0	35	0
3	9	0	12	0
2	4	1	2	1
1	1	5	1	5
0	0	117	0	117
Total	123	123	123	123

Robust Changes within Geographic Regions

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Geographic Region	Number of Countries	Robust Change in All Six Periods			
		HDI_A	Share (%)	HDI_G	Share (%)
Arab States	13	3	23.1	4	30.8
East Asia and the Pacific	17	9	52.9	8	47.1
Europe and Central Asia	27	9	33.3	9	33.3
Latin America & the Caribbean	25	5	20.0	6	24.0
North America and Oceania	4	3	75.0	3	75.0
Sub-Saharan Africa	30	2	6.7	3	10.0
South Asia	7	5	71.4	5	71.4
Total	123	36	29.3	38	30.9

Source: Author computations using UNDP data.

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- Two key objectives pursued in this paper:
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  2. Used the framework to test the robustness of improvements of the HDI over time
- Proposed robustness tests should be amenable to empirical applications

Thank you

- Questions and comments are welcome