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Geometric Mean Quantity Indices with Benefit-of-the-Doubt Weights

Tom Van Puyenbroeck
Nicky Rogge

Faculty of Economics and Business, KU Leuven

Context

- Composite Indicator Construction
- Methodological issues
  - Selection of sub-indicators
  - **commensurability**
  - **Importance in terms of contribution to composite construct (‘weighting’)**
  - **Functional form of aggregator function**
  - Robustness/sensitivity analysis of modelling options

Aggregator function

- From weighted arithmetic average to weighted geometric average

$$CI_i = \sum_{r=1}^{s} w_r y_{ri} \rightarrow CI_i = \prod_{r=1}^{s} y_{ri}^{w_r}$$

  - E.g. Human Development Index
  - More robust to rescaling (ratio-scale)
  - Diminishing marginal contribution => imperfect substitutability
  - Penalizing inequality among sub-indicators (AGM-inequality)

Benefit-of-the-doubt weighting

- Traditional model (e.g. Cherchye et al. (2004,2007)):

$$CI_i = \max_{\mathbf{w_{ri}}} \frac{\sum_{r=1}^{s} w_{ri} y_{ri}}{\max_{\mathbf{y_j} \in \{\text{studied countries}\}} \sum_{r=1}^{s} w_{ri} y_{rj}}$$

- Equivalent to LP:

$$CI_i = \max_{\mathbf{w_r}} \sum_{r=1}^{s} w_{ri} y_{ri}$$

$$s.t. \quad \sum_{r=1}^{s} w_{ri} y_{rj} \leq 1 \quad \text{(N constraints, one for each country } j = 1,..,N\text{)}$$

$$w_{ri} \geq 0 \qquad \text{(s constraints, one for each dimension } r = 1,..,s\text{)}$$

CI as index number

BoD => (Paasche) Quantity index with **shadow prices w**

$$\frac{\sum_{r=1}^{s} w_{ri} y_{ri}}{\sum_{r=1}^{s} w_{ri} y_{rj}}$$

(cf. output efficiency analysis; revenue at shadow output prices)

Multiplicative BoD-models

- Hinted at in Cherchye et al. (2007), used by Zhou et al. (2010), Blancas et al. (2012), Giambona & Vassalo (2014)

$$CI_i = \max_{w_{1i},\ldots,w_{si}} \prod_{r=1}^{s} y_{ri}^{w_{ri}}$$

$$\prod_{r=1}^{s} y_{rj}^{w_{ri}} \leq e \qquad (N \text{ constraints, one for each country } j = 1,\ldots N)$$

$$w_{ri} \geq 0 \qquad (s \text{ constraints, one for each weight } r = 1,\ldots,s)$$

Tofallis (2014)

- The preceeding model is not unit-invariant
- Alternative:

$$CI_i = \max_{w_{0i}, w_{1i}, \ldots, w_{si}} w_{0i} \prod_{r=1}^{s} y_{ri}^{w_{ri}}$$

$$\prod_{r=1}^{s} y_{rj}^{w_{ri}} \leq 1$$

$$w_{ri} \geq 1$$

- Not a geometric (weighted) average
- Alternative: pure (normalized) data
  - Other possible problems

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CI as index number

- Traditional Bod => (Paasche) Quantity index with **shadow prices**

$$\frac{\sum_{r=1}^{s} w_{ri} y_{ri}}{\sum_{r=1}^{s} w_{ri} y_{rj}}$$

- Quid multiplicative version?

$$\frac{\prod_{r=1}^{s} y_{ri}^{w_{ri}}}{\prod_{r=1}^{s} y_{rj}^{w_{ri}}} = \prod_{r=1}^{s} \left(\frac{y_{ri}}{y_{rj}} \right)^{w_{ri}}$$

Geometric mean index numbers

$$Q(p_i, y_i, p_j, y_j) = \prod_{r=1}^{s} \left(\frac{y_{ri}}{y_{rj}} \right)^{\alpha_r}$$

- $0 \leq \alpha_r \leq 1$; $\Sigma\alpha_r = 1$

- Jevons, Cobb-Douglas
  - Ad hoc (constant) weights

- Geometric Paasche, Geometric Laspeyres, Törnqvist
  - weights based on **BUDGET SHARES** (not (directly) on prices)

Back to traditional BoD

- Optimal solution yields “shadow budget shares”!
  - ‘Importance coefficients’

$$CI_i = \sum_{r=1}^{s} w_{ri}^{*} y_{ri}$$

$$\rightarrow \omega_{ri}^{*} = \frac{w_{ri}^{*} y_{ri}}{\sum_{r=1}^{s} w_{ri}^{*} y_{ri}}$$

$$(0 \leq \omega_{ri}^{*} \leq 1; \sum \omega_{ri}^{*} = 1)$$

Geometric Mean CI with BoD weights

$$CI_i^i(y_i, y_B, \omega_i^*) = \prod_{r=1}^{s} \left(\frac{y_{ri}}{y_{rB}} \right)^{\omega_{ri}^*}$$

- y_{rB} is arbitrary (e.g. sample mean, "EU-27 average",….)
- Some nice 'axiomatic' properties for an index number
  - Increasing in y_{ri}, decreasing in y_{rB}
  - Identity property
  - Linearly homogenous in comparison quantities
  - Homogenous of degree zero in quantities
  - Unit invariant

Country	CI_i^i	CI_i^i(Rank)	ω_1^*	ω_2^*	ω_3^*	ω_4^*
EU27	**1.0000**	**-**	**0.5514**	**0.3486**	**0.0500**	**0.0500**
Belgium	1.1428	17 [12]	0.7794	0.0500	0.0500	0.1206
Bulgaria	0.7367	26 [26]	0.0500	0.8500	0.0500	0.0500
Czech Republic	2.6020	2 [1]	0.0500	0.0500	0.8500	0.0500
Denmark	1.4996	11 [8]	0.6680	0.0685	0.0500	0.2135
Germany	1.1395	18 [11]	0.2738	0.5802	0.0960	0.0500
Estonia	1.0277	22 [13]	0.8149	0.0500	0.0500	0.0851
Ireland	1.0970	19 [23]	0.7759	0.0500	0.0500	0.1241
Greece	1.1956	15 [19]	0.0500	0.8500	0.0500	0.0500
Spain	0.9633	23 [21]	0.0500	0.7803	0.0500	0.1197
France	1.2360	13 [9]	0.7670	0.0692	0.0500	0.1139
Italy	1.0608	20 [17]	0.1058	0.7942	0.0500	0.0500
Cyprus	1.7409	7 [10]	0.0500	0.8500	0.0500	0.0500
Latvia	0.8943	25 [25]	0.0500	0.7556	0.1444	0.0500
Lithuania	1.5194	10 [20]	0.0500	0.0500	0.8500	0.0500
Hungary	1.2163	14 [15]	0.8500	0.0500	0.0500	0.0500
Malta	1.1602	16 [22]	0.2014	0.6986	0.0500	0.0500
Netherlands	2.1249	6 [2]	0.0500	0.5972	0.0500	0.3028
Austria	1.6468	8 [4]	0.0879	0.7964	0.0656	0.0500
Poland	2.1929	5 [14]	0.0500	0.0500	0.8500	0.0500
Portugal	1.3403	12 [18]	0.0500	0.8500	0.0500	0.0500
Romania	0.9162	24 [24]	0.0500	0.8500	0.0500	0.0500
Slovenia	2.3605	4 [5]	0.0500	0.2039	0.6961	0.0500
Slovakia	2.5404	3 [6]	0.0500	0.0500	0.8500	0.0500
Finland	1.5975	9 [7]	0.0500	0.7905	0.0605	0.0989
Sweden	5.1463	1 [3]	0.0500	0.0500	0.0500	0.8500
United Kingdom	1.0434	21 [16]	0.7329	0.0500	0.0500	0.1671
			0.2725	**0.4160**	**0.1986**	**0.1128**

1: AROP
2: % living in jobless households
3: % early school leavers
4: material deprivation rate

Multilateral versions?

- $CI(y_i, y_j, \omega) \times CI(y_j, y_k, \omega) = CI(y_i, y_k, \omega)$ for all i,j,k
- *Only if ω is common*
  - *(=> not the case with base variant)*
- **Option 1: use pivotal country (e.g. base performance data) and its associated BoD budget shares**
- **(geometric Bod Laspeyres)**

$$CI_i^B(y_i, y_B, \omega_B^*) = \prod_{r=1}^{s} \left(\frac{y_{ri}}{y_{rB}} \right)^{\omega_{rB}^*}$$

$$\frac{CI_i^B}{CI_j^B} = \frac{CI_i^B(y_i, y_B, \omega_B^*)}{CI_j^B(y_j, y_B, \omega_B^*)} = \prod_{r=1}^{s} \left(\frac{y_{ri}}{y_{rj}} \right)^{\omega_B^*}$$

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Multilateral versions?

- Option 1 is ‘weakly transitive’ in that it depends on specific choice of pivotal country
- **“base land invariance”** => CI either independent of (shadow) budget shares or a function of ALL (shadow budget shares)
- **Multilateral Generalized Törnqvist (BoD) quantity Index:**

$$M(\omega_r^*) = \frac{1}{N}\sum_{i=1}^{N}\omega_{ri}^* = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{w_{ri}^* y_{ri}}{\sum_{r=1}^{s} w_{ri}^* y_{ri}}\right)$$

$$CI_i^{MGT}(y_i, y_B, M(\omega^*)) = \prod_{r=1}^{s}\left(\frac{y_{ri}}{y_{rB}}\right)^{M(\omega_r^*)}$$

Multilateral versions?

- Recall that BoD-model applies "own" optimal weights to all countries' sub-indicator data
- "cross-efficiency notion"
  - Cross-subindicator shares $\omega_{r(j)i} = \dfrac{w_{rj}^{*} y_{ri}}{\sum_{r=1}^{s} w_{rj}^{*} y_{ri}}$
  - Mean over N cross-comparisons for country i's "peer weight based"

$$\bar{\omega}_{ri}^{cross} = \frac{1}{N}\sum_{j=1}^{n}\left(\frac{w_{rj}^{*} y_{ri}}{\sum_{r=1}^{s} w_{rj}^{*} y_{ri}}\right) = \frac{1}{N}\sum_{j=1}^{n}\omega_{r(j)i}$$

  - "grand" mean over all countries

$$\Omega_r = \frac{1}{N}\sum_{i=1}^{n}\bar{\omega}_{ri}^{cross}$$

$$CI_i^{cross}(y_i, y_B, \Omega) = \prod_{r=1}^{s}\left(\frac{y_{ri}}{y_{rB}}\right)^{\Omega_r}$$

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		ω_1^*	ω_2^*	ω_3^*	ω_4^*
	EU27-weights	0.5514	0.3486	0.0500	0.0500
	MGT-weights	0.2725	0.4160	0.1986	0.1128
	cross-weights	0.2617	0.4048	0.2081	0.1254

Country	CI_i^B	*Rank*	CI_i^{MGT}	*Rank*	CI_i^{cross}	*Rank*
EU27	1.0000	17	1.0000	16	1.0000	16
Belgium	1.0356	14	1.0390	13	1.0458	13
Bulgaria	0.7418	27	0.7007	27	0.6896	27
Czech Republic	1.6978	1	1.7548	1	1.7602	2
Denmark	1.2709	9	1.3487	8	1.3657	7
Germany	1.1152	11	1.1812	10	1.1901	10
Estonia	0.9651	19	0.9735	19	0.9765	18
Ireland	0.9064	21	0.9061	20	0.9164	20
Greece	0.9573	20	1.0123	15	1.0079	15
Spain	0.8363	23	0.8358	24	0.8389	24
France	1.1696	10	1.1521	12	1.1555	12
Italy	0.9710	18	0.9748	18	0.9746	19
Cyprus	1.2819	7	1.3170	9	1.3030	9
Latvia	0.8081	24	0.8085	25	0.7998	25
Lithuania	0.7916	26	0.8528	23	0.8529	23
Hungary	1.0150	16	0.9018	21	0.8940	22
Malta	1.0972	12	1.0292	14	1.0242	14
Netherlands	1.6821	2	1.7493	2	1.7655	1
Austria	1.4982	3	1.6055	4	1.6126	4
Poland	1.0192	15	1.1669	11	1.1693	11
Portugal	1.0518	13	0.9906	17	0.9796	17
Romania	0.8055	25	0.7647	26	0.7523	26
Slovenia	1.4935	4	1.7194	3	1.7285	3
Slovakia	1.2723	8	1.3531	7	1.3561	8
Finland	1.4045	5	1.5216	6	1.5363	6
Sweden	1.3198	6	1.5595	5	1.6034	5
United Kingdom	0.8882	22	0.9010	22	0.9107	21

Intertemporal Analysis

$$PC_i^i = \frac{CI_{i,t+1}^i}{CI_{i,t}^i} = \frac{\prod_{r=1}^{s} \left(\frac{y_{ri,t+1}}{y_{i,t+1}^B} \right)^{\omega^*_{ri,t+1}}}{\prod_{r=1}^{m} \left(\frac{y_{ri,t}}{y_{r,t}^B} \right)^{\omega^*_{ri,t}}}$$

Three components

$$PC_i^i = \prod_{r=1}^{s}\left(\frac{y_{ri,t+1}}{y_{ri,t}}\right)^{\frac{\omega^*_{ri,t+1}+\omega^*_{ri,t}}{2}} \times \prod_{r=1}^{s}\left(\frac{y^B_{r,t}}{y^B_{r,t+1}}\right)^{\frac{\omega^*_{ri,t+1}+\omega^*_{ri,t}}{2}} \times \frac{\prod_{r=1}^{s}\left(\sqrt{\frac{y_{ri,t+1}}{y^B_{r,t+1}}\frac{y_{ri,t}}{y^B_{r,t}}}\right)^{\omega^*_{ri,t+1}}}{\prod_{r=1}^{s}\left(\sqrt{\frac{y_{ri,t+1}}{y^B_{r,t+1}}\frac{y_{ri,t}}{y^B_{r,t}}}\right)^{\omega^*_{ri,t}}}$$

$$= \Delta OWN_i \times \Delta Benchmark_i \times \Delta Weight_i$$

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Country	$CI^{i}_{i,2006}$	$CI^{i}_{i,2010}$	PC^{i}_{i}	ΔOWN^{i}_{i}	ΔPB^{i}_{i}	ΔW^{i}_{i}
EU27	1.0000	1.0000	1.0000	0.9876	1.0125	1.0000
Belgium	1.1273	1.1428	1.0138	1.0198	0.9777	1.0167
Bulgaria	0.7585	0.7367	0.9712	1.0381	1.0181	0.9189
Czech Republic	2.6760	2.6020	0.9724	1.0654	0.9127	1.0000
Denmark	1.8741	1.4996	0.8001	0.8652	0.9794	0.9442
Germany	1.3070	1.1395	0.8719	0.9402	1.0032	0.9244
Estonia	1.3976	1.0277	0.7353	0.8052	1.0073	0.9065
Ireland	1.3657	1.0970	0.8033	0.9297	0.9604	0.8997
Greece	1.4897	1.1956	0.8026	0.7641	1.0503	1.0001
Spain	1.6978	0.9633	0.5674	0.6060	1.0131	0.9242
France	1.2646	1.2360	0.9774	0.9803	0.9798	1.0176
Italy	1.2782	1.0608	0.8299	0.8390	1.0295	0.9609
Cyprus	1.9058	1.7409	0.9135	0.8698	1.0503	1.0000
Latvia	1.2527	0.8943	0.7139	0.6904	1.0416	0.9928
Lithuania	1.3369	1.5194	1.1364	0.7935	0.9732	1.4715
Hungary	1.0266	1.2163	1.1847	1.2357	0.9757	0.9827
Malta	1.4573	1.1602	0.7962	0.9109	1.0062	0.8686
Netherlands	2.9202	2.1249	0.7276	1.0707	0.9491	0.7160
Austria	1.6946	1.6468	0.9718	1.0358	1.0225	0.9176
Poland	2.2863	2.1929	0.9591	1.0509	0.9127	1.0000
Portugal	1.6338	1.3403	0.8204	0.7811	1.0503	1.0000
Romania	0.8727	0.9162	1.0499	0.9985	1.0493	1.0021
Slovenia	2.4900	2.3605	0.9480	1.0492	0.9224	0.9796
Slovakia	1.6328	2.5404	1.5559	1.2644	0.9319	1.3206
Finland	1.8913	1.5975	0.8447	1.0506	0.9896	0.8124
Sweden	3.9046	5.1463	1.3180	1.5182	0.8682	1.0000
United Kingdom	1.3397	1.0434	0.7789	0.9734	0.9428	0.8487

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